Topic (Real analysis) Date 30-04-2020 B.Sc. (I)

Problem based on Taylor's theorem of two variables

Maxima and Minima of function of two variables.

Taylor's polynomial of $f(x,y)$ about the point (a,b) in powers of $(x-a)$ and $(y-b)$

$$f(x,y) = f(a,b) + [f_x^{(a,b)}(x-a) + f_y^{(a,b)}(y-b)]$$
$$+ \frac{1}{2}[f_{xx}^{(a,b)}(x-a)^2 + 2f_{xy}^{(a,b)}(x-a)(y-b) + f_{yy}^{(a,b)}(y-b)^2]$$
$$+ \frac{1}{\lfloor 3}[f_{xxx}^{(a,b)}(x-a)^3 + 3f_{xxy}(x-a)^2(y-b) + 3f_{yyx}(x-a)(y-b)^2$$
$$+ f_{yyy}(y-b)^3] + \cdots$$

Q! If $f(x,y) = x^2y + 2xy^2$ and $a=1$ $b=2$, $h=2$, $k=1$ find θ in Taylor's theorem.

$$f(a+h, b+k) = f(a,b) + (hf_x + kf_y)f(a+\theta h, b+\theta k) \quad \text{--- (1)}$$
$$0<\theta<1$$

Here $f(x,y) = x^2y + 2xy^2$

$f_x = 2xy + 2y^2 \Rightarrow hf_x = 2\times(2xy+2y^2)$

$f_y = x^2 + 4xy \Rightarrow kf_y = x^2+4xy$

$a+h = 1+2 = 3$, $b+k = 2+1 = 3$

$\therefore f[a+h, b+k] = f(3,3)$ $\quad \because f(x,y) = x^2y + 2xy^2$

$\therefore f(3,3) = 27 + 2\times3\times9$
$= 27+54 = 81$

$f(a,b) = f(1,2) = 1^2\times2 + 2\times1\times2^2$
$= 10$

$\therefore hf_x + kf_y = 4xy + 4y^2 + x^2 + 4xy$
$= x^2 + 8xy + 4y^2$

$\therefore f(a+\theta h, b+\theta k) = f(1+2\theta, 2+\theta)$

$\therefore (hf_x + kf_y)f(a+\theta h, b+\theta k) = (1+2\theta)^2 + 8(1+2\theta)(2+\theta) + 4(2+\theta)^2$
$= 1+4\theta^2+4\theta + 8(2+\theta+4\theta+2\theta^2) + 4(4+4\theta+\theta^2)$
$= 1+4\theta^2+4\theta+16+40\theta+16\theta^2+16+16\theta+4\theta^2$

Putting these values in (1) $= 24\theta^2 + 60\theta +$

$81 = 10 + 24\theta^2 + 60\theta + 33$

$\Rightarrow 24\theta^2 + 60\theta - 38 = 0$

$\Rightarrow 12\theta^2 + 30\theta - 19 = 0$

$\therefore \theta = \dfrac{-30 \pm \sqrt{900 + 4\times12\times19}}{24} = \dfrac{-15 \pm \sqrt{533}}{12}$

Since $1 < \theta < 1$, which shows only +ve sign be taken

$\theta = \dfrac{-15 + \sqrt{553}}{12} = 0.71$ approx.

Expand $f(x,y) = x^4 + x^2y^2 - y^4$ about $(1,1)$ up to terms of second degree.

Solution :→ Here

$f(x,y) = x^4 + x^2y^2 - y^4$

Since we know Expansion of $f(x,y)$ about the point (a,b) in powers of $x-a$ and $y-b$ are given as

$f(x,y) = f(a,b) + [f_x(a,b)(x-a) + f_y(a,b)(y-b)]$
$+ \frac{1}{2}[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2]$

$f(x,y) = x^4 + x^2y^2 - y^4 \Rightarrow f(1,1) = 1+1-1 = 1$

$f_x(x,y) = 4x^3 + 2xy^2 \Rightarrow f_x(1,1) = \cancel{18}\; 4+2 = 6$

$f_{xx} = 12x^2 + 2y^2 \quad f_{xx}(1,1) = 12+2 = 14$

$f_y(x,y) = 2x^2y - 4y^3 \Rightarrow f_y(1,1) = 2$

$= 2x^2y$

$f_{xy}(x,y) = 4xy \Rightarrow f_{xy}(1,1) = 4$

$f_{yy}(x,y) = 2x^2 \Rightarrow f_{yy}(1,1) = 2$

$\therefore f(x,y) = f(1,1) + [(x-1)f_x(1,1) + (y-1)f_y(1,1)]$
$+ \frac{1}{2}[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)]$

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Putting all the required values obtained as above

$$f(x,y) = 1 + \left[(x-1)\times 6 + (y-1)\times 2\right] + \frac{1}{2}\left[(x-1)^2\times 14 + 2\times 4\,(x-1)(y-1) + 2\cdot(y-1)^2\right]$$

$$= 1 + \left[6x-6+2y-2\right] + \frac{1}{2}\left[14(x-1)^2 + 8(x-1)(y-1) + 2(y-1)^2\right]$$

$$= 1 + 6x + 2y - 8 + 7(x-1)^2 + 4(x-1)(y-1) + (y-1)^2$$

$$= 1 + 6x + 2y - 8 + 7x^2 - 14x + 7 + 4(xy - x - y + 1) + y^2 - 2y + 1$$

$$= 6x + 2y - 7 + 7x^2 - 14x + 7 + 4xy - 4x - 4y + 4 + y^2 - 2y + 1$$

$$= 7x^2 + 4xy - 12x - 4y + 5 + y^2 \ldots$$

Expand $f(x,y) = e^{2x}\cos y$ as a Taylor's Series about $x=0, y=0$ as far as the terms of 3rd degree.

Solution: $\rightarrow$ Expansion of $f(x,y)$ as the 3rd degree terms about $x=a$, $y=b$

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$
$$+ \frac{1}{2}\left[f_{xx}(a,b)\cdot(x-a)^2 + 2f_{xy}(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right].$$
$$+ \frac{1}{6}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3f_{yyx}(x-a)(y-b)^2 + f_{yyy}(a,b)(y-b)^3\right] + \ldots \quad (1)$$

Here $a=0$, $b=0$

$f(x,y) = e^{2x}\cos y \Rightarrow f(0,0) = e^0\cos 0 = 1$

$f_x(x,y) = 2e^{2x}\cos y \Rightarrow f_x(0,0) = 2$

$f_y(x,y) = -e^{2x}\sin y \Rightarrow f_y(0,0) = 0$

$f_{xx}(x,y) = 4e^{2x}\cos y \Rightarrow f_{xx}(0,0) = 4$

$f_{xy}(x,y) = -2e^{2x}\sin y \Rightarrow f_{xy}(0,0) = 0$

$f_{yy} = -e^{2x}\cos y \Rightarrow f_{yy}(0,0) = -1$

$f_{xxx} = 8e^{2x}\cos y \Rightarrow f_{xxx}(0,0) = 8$

$f_{xxy} = -4e^{2x}\sin y \Rightarrow f_{xxy}(0,0) = 0$

$f_{yyx} = \cancel{+2e^{2x}\sin y} = -2e^{2x}\cos y \Rightarrow f_{yyx}(0) = -2$

$f_{yyy} = e^{2x}\sin y \therefore f_{yyy}(0,0) = 0$

Putting all these values in 1

We have

$$f(x,y) = e^{2x}\cos y = f(0,0) + [x f_x(0,0) + y f_y(0,0)]$$
$$+ \frac{1}{2}[x^2 f_{xx}(0,0) + 2 f_{xy} \cdot xy + y^2 f_{yy}(0,0)]$$
$$+ \frac{1}{\lfloor 3}[x^3 f_{xxx}(0,0) + 3 f_{xxy}(0,0)\, x^2 y + 3 f_{yyx}(0,0)\, y^2 x + f_{yyy}(0,0) \cdot y^3]$$

$$= 1 + [x \times 2 + y \cdot 0] + \frac{1}{2}[x^2 \cdot 4 + 2 \cdot 0 + y^2(-1)]$$
$$+ \frac{1}{\lfloor 3}[x^3 \cdot 8 + 3x^2 y \cdot 0 + 3xy^2(-2) + 0 \cdot y^3]$$

$$= \cancel{1 + 2x + 2x^2 - y^2}$$
$$= 1 + (x \times 2 + y \times 0) + \frac{1}{2}[x^2 \times 4 + 2xy \cdot 0 + y^2 \cdot (-1)]$$
$$+ \frac{1}{6}[x^3 \times 8 + 3x^2 y \cdot 0 + 3(-2)y^2 x + 0 \cdot y^3] + \cdots$$
$$= 1 + 2x + \frac{1}{2}[4x^2 - y^2] + \frac{1}{6}[8x^3 - 6xy^2] + \cdots$$

Ex:→ If $f(x,y) = \sqrt{|xy|}$ prove that Taylor's Expansion about the point (x,x) is not valid in any domain which includes the origin.

Here $f(x,y) = \sqrt{xy}$

$f_x(x,y) = \frac{1}{2\sqrt{x}}\sqrt{y} \Rightarrow f_x(x,y) = \frac{1}{2}\sqrt{y/x}$ when $x>0$

$= -\frac{1}{2}\sqrt{y/x}$ when $x<0$

$f_y(x,y) = \frac{\sqrt{x}}{2\sqrt{y}} \Rightarrow f_y(x,y) = \frac{1}{2}\sqrt{\frac{x}{y}}$ when $y>0$

$= -\frac{1}{2}\sqrt{x/y}$ when $y<0$

$$\therefore f_x(x,x) = f_y(x,x) = \left.\begin{array}{ll} \frac{1}{2} & x>0 \\ -\frac{1}{2} & x<0 \end{array}\right\} \longrightarrow A$$

③

Now

Taylor's theorem about (x, x), for $n=1$ is

$$f(x+h, x+h) = f(x,x) + h[f_x(x+\theta h, x+\theta h) + f_y(x+\theta h, x+\theta h)]$$

for $h = k$.

$$\Rightarrow \sqrt{(x+h)(x+h)} = \sqrt{x \cdot x} + h[f_x(x+\theta h, x+\theta h) + f_y(x+\theta h, x+\theta h)]$$

$$\Rightarrow |x+h| = |x| + h[f_x(x+\theta h, x+\theta h) + f_y(x+\theta h, x+\theta h)] \quad \text{— (1)}$$

If $x+\theta h > 0$ from (A)

$$f_x(x+\theta h, x+\theta h) = f_y(x+\theta h, x+\theta h) = 1/2$$

$\therefore$ From (1)

$$|x+h| = |x| + h(1/2 + 1/2) = |x| + h$$

Similarly if $x+\theta h < 0$

then $|x+h| = x - h$

and if $x+\theta h = 0$ then $|x+h| = |x| + h[0-0] = |x|$

Thus
$$|x+h| = \begin{cases} |x|+h & x+\theta h > 0 \\ |x|-h & x+\theta h < 0 \\ |x| & x+\theta h = 0 \end{cases} \quad \text{— (B)}$$

Now if the domain $(x, x, x+h, x+h)$ includes origin

then x and $x+h$ must be opposite sign

i.e if $x+h > 0$ then $x < 0$ and hence

$|x+h| = x+h$ and $|x| = x$

and if $x+h < 0$ then $|x| > 0$ and hence $|x+h| = -(x+h)$ and $|x| = x$

But under these conditions, of the inequalities (B) holds.

this expansion is not valid. None of the